Effects of Finite Larmor Radius and Hall Currents on Thermosolutal Instability of a Partially Ionized Plasma in Porous Medium

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The effects of finite ion Larmor radius (FLR), collisions and Hall currents on thermosolutal instability of a partially ionized plasma in porous medium in the presence of uniform vertical magnetic field are investigated. It is found that the presence of each magnetic field, FLR, Hall currents and collisions, introduces oscillatory modes which were, otherwise, non-existent. In the case of stationary convection, finite Larmor radius, Hall currents, medium permeability and magnetic field may have stabilizing or destabilizing effects, but for a certain wave number range, FLR, magnetic field and Hall currents have a complete stabilizing effect. The stable solute gradient always has stabilizing effect on the system whereas the collisional effects disappear for the case of stationary convection.

1. Introduction

The theoretical and experimental results on thermal convection in a fluid layer under varying assumptions of hydrodynamics and hydromagnetics have been discussed in a treatise by Chandrasekhar [1]. The stabilizing influence of finite Larmor radius (FLR) effects on thermal instability, thermosolutal instability, gravitational instability and Rayleigh-Taylor instability has individually been shown by several authors [2–6].

The thermohaline convection in a horizontal layer of a viscous fluid heated from below and salted from above has been considered by Nield [7]. In the thermohaline convection, buoyancy forces can arise not only from density differences due to variations in temperature but also from those due to variations in salt concentration. The problem of the onset of thermal instability in the presence of a solute gradient is of great importance because of its applications to atmospheric physics and astrophysics, especially in the case of the ionosphere and the outer layers of the solar atmosphere (Spiegel [8]). The Hall currents are likely to be important in these regions. Sharma and Rani [9] have considered the effect of Hall currents on the thermosolutal instability of a plasma.

A partially-ionized plasma represents a state which often exists in the universe, and there are several situations when the interaction between ionized and neutral gas components becomes important in cosmic physics. Strömgren [10] has reported that ionized hydrogen is limited to certain rather sharply bounded regions in space surrounding, for example O-type stars and clusters of such stars, and that the gas outside these regions is specially non-ionized. Piddington [11] and Lehnert [12] have found that both ion viscosity and neutral gas friction have a stabilizing influence on cosmical plasma interacting with neutral gas.

In recent years, there has been a considerable interest in the study of the breakdown of the stability of a layer of fluid subject to a vertical temperature gradient in a porous medium and the possibility of convective flow. The physical properties of comets, meteorites and interplanetary dust strongly suggest the importance of porosity in astrophysical context (McDonnel [13]). The stability of flow of a single component fluid through a porous medium taking into account the Darcy resistance has been studied by Lapwood [14] and Wooding [15].

Seeing the importance of Hall currents, finite ion Larmor radius, partially-ionized plasma and medium porosity in astrophysical situations like stellar interiors and atmospheres in geophysical situations like the Earth's molten core, in flows of laboratory plasma and in the stellar atmosphere where helium acts like a salt in raising the density and in diffusing more slowly than heat, in the present paper we discuss the thermosolutal instability of a partially-ionized plasma in po-

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rous medium in the presence of a uniform vertical magnetic field to include simultaneously the effects of collisions, FLR and Hall currents.

2. Perturbation Equations and Dispersion Relation

Consider an incompressible, viscous, infinite, horizontal, composite plasma layer of depth d, consisting of an infinitely electrical conducting ionized component of density ϱ, a neutral component of density ϱ_d and bounded by the planes $z=0$ and $z=d$ in an isotropic and homogeneous medium of porosity ε and permeability k_1. This layer is heated and soluted from below such that a uniform temperature gradient $\beta\left(=\left|\frac{\mathrm{d}T}{\mathrm{d}z}\right|\right)$ and a uniform solute concentration gradient $\beta'\left(=\left|\frac{\mathrm{d}C}{\mathrm{d}z}\right|\right)$ are maintained, where T and C denote the temperature and solute concentration, respectively. The system is acted on by a uniform vertical magnetic field $\boldsymbol{H}(0,0,H)$ and gravity force $\boldsymbol{g}(0,0,-g)$. Let p, α, α', η, N, e, ν_c and $\vec{\mathrm{P}}$ denote, respectively, the pressure, thermal coefficient of expansion, an analogous solvent coefficient of expansion, resistivity, electron number density, charge of an electron, mutual collisional frequency between the two components of the composite medium, and stress tensor taking into account the FLR effect. The magnetic permeability μ_e, the kinematic viscosity $\nu\left(=\frac{\mu}{\varrho_0}\right)$, the thermal diffusivity $\varkappa$, and the solute diffusivity $\varkappa'$ are all assumed to be constant. We assume that both the ionized component and the neutral component behave like a continuum plasma and that the effects on the neutral component resulting from the presence of magnetic field, porosity, and the fields of gravity and pressure are neglected.

Let δp, $\delta\varrho$, θ, γ, $\boldsymbol{h}(h_x,h_y,h_z)$, $\boldsymbol{q}(u,v,w)$, and $\boldsymbol{q}_d(l,r,s)$ denote, respectively, the perturbations in pressure p, density ϱ, temperature T, solute concentration C, magnetic field $\boldsymbol{H}$, ionized component velocity (initially zero), and neutral component velocity (initially zero). ϱ_d is assumed to be constant everywhere. Then the linearized hydromagnetic perturbation equations of the composite medium are

$$\frac{1}{\varepsilon}\frac{\partial \boldsymbol{q}}{\partial t} = -\frac{1}{\varrho_0}\nabla\delta p+\boldsymbol{g}\frac{\delta\varrho}{\varrho_0}+\frac{\mu_e}{4\pi\varrho_0}(\nabla\times\boldsymbol{h})\times\boldsymbol{H}-\frac{1}{\varrho_0}\nabla\vec{\mathrm{P}}+\frac{\varrho_d\nu_c}{\varrho_0\varepsilon}(\boldsymbol{q}_d-\boldsymbol{q})-\frac{\nu}{k_1}\boldsymbol{q}, \tag{1}$$

$$\frac{\partial \boldsymbol{q}_d}{\partial t} = -\nu_c(\boldsymbol{q}_d-\boldsymbol{q}), \tag{2}$$

$$E\frac{\partial\theta}{\partial t} = \beta w+\varkappa\nabla^2\theta, \tag{3}$$

$$E'\frac{\partial\gamma}{\partial t} = \beta' w+\varkappa'\nabla^2\gamma, \tag{4}$$

$$\varepsilon\frac{\partial \boldsymbol{h}}{\partial t} = (\boldsymbol{H}\cdot\nabla)\boldsymbol{q}+\varepsilon\eta\nabla^2\boldsymbol{h}-\frac{c'\varepsilon}{4\pi Ne}\nabla\times[(\nabla\times\boldsymbol{h})\times\boldsymbol{H}], \tag{5}$$

$$\nabla\cdot\boldsymbol{h}=0 \text{ and } \nabla\cdot\boldsymbol{q}=0, \tag{6}$$

where c' denotes the speed of light, $E=\varepsilon+(1-\varepsilon)\frac{\varrho_s c_s}{\varrho_0 c_i}$ is a constant, and E' is a constant analogous to E but corresponding to solute rather than heat. $\varrho_s, c_s; \varrho_0, c_i$ denote the density and heat capacity of the solid matrix and ionized component, respectively.

The change in density $\delta\varrho$, caused by the perturbation θ and γ in temperature and concentration, is given by

$$\delta\varrho = -\varrho_0(\alpha\theta-\alpha'\gamma). \tag{7}$$

For the magnetic field along z-axis, the stress tensor $\vec{\mathrm{P}}$, taking into account the finite ion-gyration radius (Vandakurov [16]), has the components

$$\begin{aligned}
P_{xx}&=-\varrho_0\nu_0\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right),\\
P_{xy}&=P_{yx}=\varrho_0\nu_0\left(\frac{\partial u}{\partial x}-\frac{\partial v}{\partial y}\right),\\
P_{xz}&=P_{zx}=-2\varrho_0\nu_0\left(\frac{\partial v}{\partial z}+\frac{\partial w}{\partial y}\right),\\
P_{yy}&=\varrho_0\nu_0\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right),\\
P_{yy}&=\varrho_0\nu_0\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right),\\
P_{yz}&=P_{zy}=2\varrho_0\nu_0\left(\frac{\partial w}{\partial x}+\frac{\partial u}{\partial z}\right),\\
P_{zz}&=0,
\end{aligned} \tag{8}$$

where $\varrho_0\nu_0=\frac{NT}{4\omega_H}$, ω_H being the ion-gyration frequency while N and T are the number density and temperature of ions, respectively.

Analyzing the disturbances of normal modes, we assume that the perturbation quantities are of the form

$$[w, \theta, \gamma, h_z, \zeta, \xi] = [W(z), \Theta(z), \Gamma(z), K(z), Z(z), X(z)] \times \exp[i k_x x + i k_y y + n t], \quad (9)$$

where k_x and k_y are the wave numbers along the x- and y-directions, respectively. $k = (k_x^2 + k_y^2)^{1/2}$ is the resultant wave number and n is the growth rate of disturbance. $\zeta = i k_x v - i k_y u$ and $\xi = i k_x h_y - i k_y h_x$ denote, respectively, the z-components of vorticity and current density.

Using (7), (8) and expression (9), (1)–(6) in non-dimensional form become

$$\left[\sigma\left(1+\frac{\alpha_0 v_c'}{\sigma+v_c'}\right)+\frac{1}{P_l}\right](D^2-a^2)W+\frac{g d^2 a^2 \varepsilon}{v}(\alpha\Theta-\alpha'\Gamma) + \frac{v_0 \,\mathrm{d}\varepsilon}{v}(2D^2+a^2)DZ-\frac{\mu_e H \alpha \varepsilon}{4\pi \varrho_0 v}(D^2-a^2)DK=0, \quad (10)$$

$$[\sigma\left(1+\frac{\alpha_0 v_c'}{\sigma+v_c'}\right)+\frac{1}{P_l}\Big] Z=\frac{v_0\, \varepsilon}{v_d}(2D^2+a^2)DW + \frac{\mu_e H d \varepsilon}{4\pi\varrho_0 v} DX, \quad (11)$$

$$(D^2-a^2-p_2\sigma)K=-\left(\frac{H d}{\eta\varepsilon}\right)DW + \frac{c' H d}{4\pi N e \eta} DX, \quad (12)$$

$$(D^2-a^2-p_2\sigma)X=-\left(\frac{H d}{\eta\varepsilon}\right)DZ - \frac{c' H}{4\pi N e \eta d}(D^2-a^2)DK, \quad (13)$$

$$(D^2-a^2-E p_1\sigma)\Theta=-\left(\frac{\beta d^2}{\varkappa}\right)W, \quad (14)$$

$$(D^2-a^2-E' q\sigma)\Gamma=-\left(\frac{\beta' d^2}{\varkappa'}\right)W, \quad (15)$$

where we have put $x=x^* d$, $y=y^* d$, $z=z^* d$ and $D=\frac{\mathrm{d}}{\mathrm{d}z^*}$ in the new unit of length d (omitting the asterisks for simplicity) and letting $a=kd$, $\sigma=\frac{n d^2}{v}$, $v_c'=\frac{v_c d^2}{v}$, $p_1=\frac{v}{\varkappa}$, $p_2=\frac{v}{\eta}$, $q=\frac{v}{\varkappa'}$, $P_l=\frac{k_1}{\varepsilon d^2}$.

We consider the case in which both boundaries are free, the most appropriate case for a stellar atmosphere (Spiegel [17]), and the adjoining medium is electrically non-conducting. The appropriate boundary conditions for this case, by use of (9), are

$$W=D^2 W=X=DZ=\Theta=\Gamma=0 \text{ at } z=0 \text{ and } z=d \quad (16)$$

and $\boldsymbol{h}$ is continuous.

Since the components of the magnetic field are continuous, and tagential components are zero outside the plasma, we get

$$DK=0 \quad (17)$$

on the boundaries.

Eliminating Θ, Γ, K, Z and X in (10)–(15) and using the proper solution $W=W_0 \sin\pi z$, W_0 being constant, we obtain the following dispersion relation:

$$\begin{aligned} R_1 = {} & S_1(1+x+iEp_1\sigma_1)(1+x+E'q\sigma_i)^{-1} \\ & +\Bigg[(1+x)\left\{i\sigma_1\left(1+\frac{\alpha_0 v_c'}{i\sigma_1\pi^2+v_c'}\right)+\frac{1}{P}\right\}^2(1+x+iEp_1\sigma_1) \\ & \times\{(1+x+ip_2\sigma_1)^2+M(1+x)\}+Q_1(1+x)(1+x+iEp_1\sigma_1) \\ & \times\left\{\left(2i\sigma_1\left(1+\frac{\alpha_0 v_c'}{i\sigma_1\pi^2+v_c'}\right)+\frac{2}{P}\right)(1+x+ip_2\sigma_1)+Q_1\right\} \\ & +U(x-2)^2(1+x+iEp_1\sigma_1)\{(1+x+ip_2\sigma_1)^2+M(1+x)\} \\ & +2U^{1/2}M^{1/2}Q_1(1+x)(x-2)(1+x+iEp_1\sigma_1)\Bigg] \\ & \times\varepsilon^{-1}x^{-1}\left[\{(1+x+ip_2\sigma_1)^2+M(1+x)\}\times\left\{i\sigma_1\left(1+\frac{\alpha_0 v_c'}{i\sigma_1\pi^2+v_c'}\right)+\frac{1}{P}\right\}+Q_1(1+x+ip_2\sigma_1)\right]^{-1}, \end{aligned} \quad (18)$$

where

$$x=\frac{a^2}{\pi^2},\ i\sigma_1=\frac{\sigma}{\pi^2},\ P=\pi^2 P_l,\ U=\frac{\nu_0^2\,\varepsilon^2}{\nu^2},$$

$$Q=\frac{\mu_e H^2 d^2}{4\pi\varrho_0\nu\eta}\ \text{(Chandrasekhar number)},\ Q_1=\frac{Q}{\pi^2},$$

$$R=\frac{g\alpha\beta d^4}{\nu\varkappa}\ \text{(Rayleigh number)},\qquad R_1=\frac{R}{\pi^4},$$

$$S=\frac{g\alpha'\beta' d^4}{\nu\varkappa'}\ \text{(analogous solute Rayleigh number)},\qquad S_1=\frac{S}{\pi^4},$$

and $M=\left(\frac{c'H}{4\pi N e\eta}\right)^2$ is a non-dimensional number accounting for Hall currents.

3. The Stationary Convection

When the instability sets in as stationary convection, the marginal state will be characterized by $\sigma=0$. In this case (18) reduces to

$$R_1=S_1+\left(\frac{1+x}{\varepsilon x}\right)\times\frac{\left[\frac{1+x}{\varrho}+Q_1\right]^2+\frac{M(1+x)}{P^2}+U(x-2)^2(1+x+M)+2Q_1U^{1/2}M^{1/2}(x-2)}{\left(\frac{1+x+M}{P}+Q_1\right)}. \tag{19}$$

Equation (19) expresses R_1 as a function of the dimensionless wave number x and the parameters S_1, Q_1, U, M, and P. To study the effects of stable solute gradient, finite Larmor radius, Hall currents, magnetic field and medium permeability, we examine the nature of

$$\frac{dR_1}{dS_1},\ \frac{dR_1}{dU},\ \frac{dR_1}{dM},\ \frac{dR_1}{dQ_1},\ \text{and}\ \frac{dR_1}{dP}$$

analytically.

From (19) follows

$$\frac{dR_1}{dS_1}=+1, \tag{20}$$

which implies that the stable solute gradient has stabilizing effect on the system.

$$\frac{dR_1}{dU}=\left(\frac{1+x}{\varepsilon x}\right)\times\frac{(x-2)^2(1+x+M)+Q_1M^{1/2}U^{-1/2}(x-2)}{\left[\frac{1+x+M}{P}+Q_1\right]}, \tag{21}$$

which is positive if $x>2$.

Thus the FLR stabilizes the system for the wave number range $x>2$. In the absence of Hall currents the FLR always has a stabilizing effect on the thermosolutal instability. Also (19) yields

$$\frac{dR_1}{dM}=\left(\frac{1+x}{\varepsilon x}\right)\times\frac{\left[Q_1\left\{\sqrt{\frac{U}{M}}(x-2)-\frac{1}{P}\right\}\right]\times\left[\frac{1+x}{P}+\sqrt{UM}(x-2)+Q_1\right]}{\left[\frac{1+x+M}{P}+Q_1\right]^2}, \tag{22}$$

$$\frac{dR_1}{dM}>0\ \text{if}\ x>2\ \text{and}\ \sqrt{\frac{U}{M}}(x-2)>\frac{1}{P}. \tag{23}$$

The Hall currents, therefore, may have stabilizing or destabilizing effect but have a stabilizing effect for the wave number range (23). In the absence of FLR effects, the Hall currents have a destabilizing effect on the thermosolutal instability of a partially-ionized plasma in porous medium. Again

$$\frac{dR_1}{dQ_1}=\left(\frac{1+x}{\varepsilon x}\right)\times\frac{\left[\frac{2Q_1}{P}(1+x+M)+Q_1^2+\left(\frac{1+x}{P}\right)(1+x+M)+(1+x+M)\left\{\frac{2\sqrt{UM}}{P}(x-2)-U(x-2)^2\right\}\right]}{\left[\frac{1+x+M}{P}+Q_1\right]^2}. \tag{24}$$

If $x>2$ and $\sqrt{\frac{U}{M}}(x-2)<\frac{2}{P}$, (25)

$\frac{dR_1}{dQ_1}>0$, exhibiting that the magnetic field completely stabilizes the system for the above wave number range given by (25). Out of this range, the magnetic field may have a stabilizing or destabilizing effect. Also

$$\frac{dR_1}{dP}=-\left(\frac{1+x}{\varepsilon x P^2}\right)\times\left(\frac{1+x+M}{P}+Q_1\right)^{-2}$$
$$\times\Bigg[(1+x)\left(\frac{1+x+M}{P}\right)^2+\frac{2Q_1}{P}(1+x) \tag{26}$$
$$\times(1+x+M)+Q_1^2(1+x-M)-(1+x+M)$$
$$\times\left\{U(x-2)^2(1+x+M)+2Q_1\sqrt{UM}(x-2)\right\}\Bigg],$$

which may be positive (or negative) under certain conditions exhibiting the stabilizing (or destabilizing) effect of medium permeability in the presence of magnetic field, FLR effects and Hall currents. In the absence of magnetic field, FLR effects and Hall currents

$$\frac{dR_1}{dP}<0, \tag{27}$$

the medium permeability always has a destabilizing effect on the system.

Also we notice from (19) that there is no collisional term. So collisional effects disappear in the case of stationary convection.

4. The Oscillatory Models

Here we examine the possibility of oscillatory modes, if any, on the stability problem due to the presence of magnetic field, finite Larmor radius and Hall currents. Multiplying (10) by W^*, the complex conjugate of W, integrating over the range of z and making use of (11)–(15) together with boundary conditions (16) and (17) we obtain,

$$\left[\sigma\left(1+\frac{\alpha_0 v_c'}{\sigma+v_c'}\right)+\frac{1}{P_l}\right]I_1-\frac{g\alpha\varkappa a^2\varepsilon}{v\beta}[I_2+Ep_1\sigma^* I_3]$$
$$+\frac{g\alpha'\varkappa' a^2\varepsilon}{v\beta'}[I_4+E'q\sigma^* I_5]+\frac{\mu_e\eta\varepsilon^2}{4\pi\varrho_0 v}[I_6+p_2\sigma^* I_7]$$
$$+\frac{\mu_e\eta\varepsilon^2 d^2}{4\pi\varrho_0 v}[I_8+p_2\sigma I_9] \tag{28}$$
$$+d^2\left[\sigma^*\left(1+\frac{\alpha_0 v_c'}{\sigma^*+v_c'}\right)+\frac{1}{P_l}\right]I_6=0,$$

where σ^* is the complex conjugate of σ and

$$\begin{aligned}
I_1&=\int_0^1(|DW|^2+a^2|W|^2)\,dz,\\
I_2&=\int_0^1(|D\Theta|^2+a^2|\Theta|^2)\,dz,\\
I_3&=\int_0^1|\Theta|^2\,dz,\\
I_4&=\int_0^1(|D\Gamma|^2+a^2|\Gamma|^2)\,dz,\\
I_5&=\int_0^1|\Gamma|^2\,dz,\\
I_6&=\int_0^1(|D^2K|^2+2a^2|DK|^2+a^4|K|^2)\,dz,\\
I_7&=\int_0^1(|DK|^2+a^2|K|^2)\,dz,\\
I_8&=\int_0^1(|DX|^2+a^2|X|^2)\,dz,\\
I_9&=\int_0^1|X|^2\,dz,\\
I_{10}&=\int_0^1|Z|^2\,dz.
\end{aligned} \tag{29}$$

The integrals I_1-I_{10} are all positive definite. Substituting $\sigma=i\sigma_0$, where σ_0 is real, in (28) and equating the imaginary parts, we obtain

$$\sigma_0\Bigg[\frac{\sigma_0^2+(1+\alpha_0)v_c'^2}{v_c'^2+\sigma_0^2}(I_1-d^2I_6)+\frac{g\alpha\varkappa a^2\varepsilon}{v\beta}$$
$$\times Ep_1I_3-\frac{g\alpha'\varkappa' a^2\varepsilon}{v\beta'}E'qI_5-\frac{\mu_e\eta\varepsilon^2}{4\pi\varrho_0 v}p_2I_7 \tag{30}$$
$$+\frac{\mu_e\eta\varepsilon^2 d^2}{4\pi\varrho_0 v}p_2I_9\Bigg]=0.$$

Equation (30) yields that $\sigma_0=0$ or $\sigma_0\neq 0$, which means that modes may be non-oscillatory or oscillatory. In the absence of magnetic field, Larmor radius and Hall currents, (30) reduces to

$$\sigma_0\left[I_1+\frac{g\alpha\varkappa a^2\varepsilon}{v\beta}Ep_1I_3\right]=0, \tag{31}$$

and the terms in the brackets are positive definite.

Thus $\sigma_0=0$, which means the modes are non-oscillatory and the principle of exchange of stabilities is satisfied. The presence of each: magnetic field, finite Larmor radius, collisions and Hall currents, introduces oscillatory modes which were, otherwise, non-existent.

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